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# Quantum Deformation of the Lie Superalgebra *spl*(2, 1)

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In this paper we obtain a q-deformation of the Lie superalgebra *spl*(2, 1), which is in correspondence with its one-parameter quantum group. This procedure suggests a two-parameter deformation of the Lie superalgebra which leads to the two-parameter quantum group.

Recently a great deal of attention has been paid to the algebraic structure called a quantum group; this algebraic structure was created by Jimbo and Drinfeld [1,2]. The one parameter quantum groups are introduced as one-parameter deformations of the universal enveloping algebra $U(L)$ of an algebra $L$ leading to noncommutative and noncocommutative Hopf algebra $U_q(L)$, namely the quantum groups; quantum groups would be also considered as a nontrivial generalization of the ordinary Lie groups. (If $L$ is a Lie algebra, we deal with quantum groups and if $L$ is a Lie superalgebra we deal with quantum supergroups [3–5]). One of the simplest examples of a quantum group is $SU_q(2)$ and its realization, which has been obtained by Biedenharn and Macfarlane [6,7]. The q-deformation of the supersymmetric oscillator including the q-creation and q-annihilation operators is considered in, e.g., refs. 3, 4, 9, and 10.

The two parameter $R_{p,q}$-matrix given in ref. 13 of the two-parameter quantum group has been extensively used [11,12], where $R_{p,q} = R_q \cdot T$,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (1/q - 1/p) & qp^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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with the Hecke relation

$$(\sigma_{p,q} + 1)(\sigma_{p,q} - 1/pq) = 0; \qquad \sigma_{p,q} = p^{-1} R_{p,q}^{-1}$$

where $R_q$ is the $R$-matrix of the one-parameter group [14].

Let us consider the Lie superalgebra $spl(2, 1)$ whose even part is $sl(2) \otimes gl(1)$; the generators $J_3$, $J_\pm$ of $sl(2)$ are called isospin and the generator $B$ of $gl(1)$ is called baryon number. The odd generators $V_\pm$ and $W_\pm$ carry baryon number $+1/2$ and $-1/2$, resp.; they are $sl(2)$ spinors. The Lie superalgebra $spl(2, 1)$ is represented by eight generators $J_\pm$, $V_\pm$, $W_\pm$, $J_3$, and $B$ such that

$$\begin{aligned}
&[J_3, J_\pm] = \pm J_\pm, && [J_+, J_-] = 2J_3\\
&[B, J_\pm] = [B, J_3] = 0, && [J_3, V_\pm] = \pm\tfrac{1}{2} V_\pm\\
&[B, V_\pm] = \tfrac{1}{2} V_\pm, && [B, W_\pm] = -\tfrac{1}{2} W_\pm\\
&[J_\pm, V_\mp] = V_\pm, && [J_\pm, W_\mp] = W_\pm\\
&[J_3, W_\pm] = \pm\tfrac{1}{2} W_\pm, && [J_\pm, V_\pm] = [J_\pm, W_\pm] = 0\\
&\{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = 0, && \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0\\
&\{V_\pm, W_\pm\} = \pm J_\pm, && \{V_\pm, W_\mp\} = -J_3 \pm B
\end{aligned}$$

We assume that

$$[J_3, V_\pm] = \pm\tfrac{1}{2} V_\pm \tag{1}$$

$$[J_3, W_\pm] = \pm\tfrac{1}{2} W_\pm \tag{2}$$

$$\{V_+, W_+\} = f(q) J_+ \tag{3}$$

$$\{V_-, W_-\} = g(q) J_- \tag{4}$$

where $f(q)$ and $g(q)$ are functions of $q$.

We can now prove that our assumptions is consistent with the commutation relation of the Lie superalgebra $spl(2, 1)$, that is,

$$[J_3, J_+] = J_+ \tag{5}$$

$$[J_3, J_-] = -J_- \tag{6}$$

Consider now the following assumption:

$$\{V_+, W_-\} = -F(J_3) + B \tag{7}$$

$$\{V_-, W_+\} = -F(J_3) - B \tag{8}$$

where $F(J_3)$ is an arbitrary function of $J_3$, then we deduce that

$$[J_+, V_-] = \frac{2}{f(q)}\left\{F(J_3) - F\left(J_3 - \frac{1}{2}\right)\right\}V_+ \tag{9}$$

$$[J_-, V_+] = \frac{2}{g(q)}\left\{F(J_3) - F\left(J_3 + \frac{1}{2}\right)\right\}V_- \tag{10}$$

$$[J_+, W_-] = \frac{2}{f(q)}\left\{F(J_3) - F\left(J_3 - \frac{1}{2}\right)\right\}W_+ \tag{11}$$

$$[J_-, W_+] = \frac{2}{g(q)}\left\{F(J_3) - F\left(J_3 + \frac{1}{2}\right)\right\}W_- \tag{12}$$

$$[B, V_+] = \left\{F(J_3) - F\left(J_3 - \frac{1}{2}\right)\right\}V_+ \tag{13}$$

$$[B, V_-] = \left\{F(J_3) - F\left(J_3 + \frac{1}{2}\right)\right\}V_- \tag{14}$$

$$[B, W_+] = -\left\{F(J_3) - F\left(J_3 - \frac{1}{2}\right)\right\}W_+ \tag{15}$$

$$[B, W_-] = \left\{F(J_3) - F\left(J_3 + \frac{1}{2}\right)\right\}W_- \tag{16}$$

$$\begin{aligned}[J_+, J_-] = \frac{1}{g(q)f(q)}\Bigg\{&\left[B - F(J_3) + 2F\left(J_3 + \frac{1}{2}\right)\right]W_-V_+ \\ &+ \left[-B - F(J_3) + 2F\left(J_3 + \frac{1}{2}\right)\right]V_-W_+ \\ &- \left[B - F(J_3) + 2F\left(J_3 - \frac{1}{2}\right)\right]V_+W_- \\ &+ \left[B + F(J_3) - 2F\left(J_3 - \frac{1}{2}\right)\right]W_+V_-\Bigg\}\end{aligned} \tag{17}$$

Now we define the following three types of $q$-number:

$$[x] = \frac{q^{x/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}} \tag{18}$$

$$[x]_q = \frac{q^{x} - q^{-x}}{q - q^{-1}} \tag{19}$$

$$[x]_+ = \frac{q^{x/2} + q^{-x/2}}{q^{1/2} + q^{-1/2}} \tag{20}$$

where $[x]_+[x] = [x]_q$.

If we choose $F(J_3)$ as

$$F(J_3) = [J_3]_q \tag{21}$$

we obtain

$$[J_+, V_-] = \frac{2}{f(q)}\left[\frac{1}{2}\right]\left[2J_3 - \frac{1}{2}\right]_+ V_+ \tag{22}$$

$$[J_-, V_+] = \frac{-2}{g(q)}\left[\frac{1}{2}\right]\left[2J_3 + \frac{1}{2}\right]_+ V_- \tag{23}$$

$$[J_+, W_-] = \frac{2}{f(q)}\left[\frac{1}{2}\right]\left[2J_3 - \frac{1}{2}\right]_+ W_+ \tag{24}$$

$$[J_-, W_+] = \frac{-2}{g(q)}\left[\frac{1}{2}\right]\left[2J_3 + \frac{1}{2}\right]_+ W_- \tag{25}$$

$$[B, V_+] = \left[\frac{1}{2}\right]\left[2J_3 - \frac{1}{2}\right]_+ V_+ \tag{26}$$

$$[B, V_-] = \left[\frac{1}{2}\right]\left[2J_3 + \frac{1}{2}\right]_+ V_- \tag{27}$$

$$[B, W_+] = -\left[\frac{1}{2}\right]\left[2J_3 - \frac{1}{2}\right]_+ W_+ \tag{28}$$

$$[B, W_-] = -\left[\frac{1}{2}\right]\left[2J_3 + \frac{1}{2}\right]_+ W_- \tag{29}$$

$$[J_+, J_-] = \frac{1}{g(q)f(q)} \{(B - [J_3]_q + 2[J_3 + 1/2]_q)W_-V_+ + (-B - [J_3]_q + 2[J_3 + 1/2]_q)V_-W_+ - (B - [J_3]_q + 2[J_3 - 1/2]_q)V_+W_- + (B + [J_3]_q - 2[J_3 - 1/2]_q)W_+V_-\} \tag{30}$$

Now when $q$ tends to 1 equation (22) becomes

$$[J_+, V_-] = \frac{1}{f(1)} V_+$$

but $[J_+, V_-] = V_+$; then $f(1) = 1$, and equation (23) becomes

$$[J_-, V_+] = \frac{-1}{g(1)} V_-$$

but $[J_-, V_+] = V_-$, so we deduce that $g(1) = -1$. We can choose $f(q)$ and $g(q)$ as follows:

$$f(q) = \frac{2}{q^{1/2} + q^{-1/2}}, \qquad g(q) = \frac{-2}{q^{1/2} + q^{-1/2}} \tag{31}$$

or any other form such that $f(q) \to 1$ as $q \to 1$ and $g(q) \to -1$ as $q \to 1$. Then we have

$$[J_+, V_-] = (q^{1/2} + q^{-1/2})\left[\frac{1}{2}\right]\left[2J_3 - \frac{1}{2}\right]_+ V_+ \tag{32}$$

$$[J_-, V_+] = (q^{1/2} + q^{-1/2})\left[\frac{1}{2}\right]\left[2J_3 + \frac{1}{2}\right]_+ V_- \tag{33}$$

$$[J_+, W_-] = (q^{1/2} + q^{-1/2})\left[\frac{1}{2}\right]\left[2J_3 - \frac{1}{2}\right]_+ W_+ \tag{34}$$

$$[J_-, W_+] = (q^{1/2} + q^{-1/2})\left[\frac{1}{2}\right]\left[2J_3 + \frac{1}{2}\right]_+ W_- \tag{35}$$

$$[B, V_+] = \left[\frac{1}{2}\right]\left[2J_3 - \frac{1}{2}\right]_+ V_+ \tag{36}$$

$$[B, V_-] = \left[\frac{1}{2}\right]\left[2J_3 + \frac{1}{2}\right]_+ V_- \tag{37}$$

$$[B, W_+] = -\left[\frac{1}{2}\right]\left[2J_3 - \frac{1}{2}\right]_+ W_+ \tag{38}$$

$$[B, W_-] = -\left[\frac{1}{2}\right]\left[2J_3 + \frac{1}{2}\right]_+ W_- \tag{39}$$

$$\begin{aligned}[J_+, J_-] = &-\left(\frac{q^{1/2} + q^{-1/2}}{2}\right)^2 \{(B - [J_3]_q + 2[J_3 + 1/2]_q)W_-V_+ \\ &+ (-B - [J_3]_q + 2[J_3 + 1/2]_q)V_-W_+ \\ &- (B - [J_3]_q + 2[J_3 - 1/2]_q)V_+W_- \\ &+ (B + [J_3]_q - 2[J_3 - 1/2]_q)W_+V_-\}\end{aligned} \tag{40}$$

Other $q$-deformation are given when we choose $F(J_3)$ in another form such that $F(J_3) \to J_3$ as $q \to 1$ [14].

The two-parameter quantum deformation is given by considering that $f = f(p, q)$ and $g = g(p, q)$, where $p$, $q$ are independent arbitrary parameters. We define the $(p, q)$ number

$$[x]_{p,q} = \frac{q^{2x} - q^{-2x}}{(p^x + p^{-x})(q - q^{-1})} \tag{41}$$

such that $[x]_{p,q} \to [x]_q$ as $p = q$; as convention we use

$$[(x)_p] = [x], \qquad \text{when } p \text{ is the parameter}$$
$$[(x)_q] = [x], \qquad \text{when } q \text{ is the parameter}$$

then we have

$$[J_+, V_-] = \frac{2}{f(p, q)} \frac{[2J_3]_q[(2J_3 - 1)_p]_+ - [(2J_3)_p]_+[2J_3 - 1]_q}{1 + [(4J_3 - 1)_p]_+} V_+ \tag{42}$$

$$[J_-, V_+] = \frac{2}{g(p, q)} \frac{[2J_3]_q[(2J_3 + 1)_p]_+ - [(2J_3)_p]_+[2J_3 + 1]_q}{1 + [(4J_3 + 1)_p]_+} V_- \tag{43}$$

$$[J_+, W_-] = \frac{2}{f(p, q)} \frac{[2J_3]_q[(2J_3 - 1)_p]_+ - [(2J_3)_p]_+[2J_3 - 1]_q}{1 + [(4J_3 - 1)_p]_+} W_+ \tag{44}$$

$$[J_-, W_+] = \frac{2}{g(p, q)} \frac{[2J_3]_q[(2J_3 + 1)_p]_+ - [(2J_3)_p]_+[2J_3 + 1]_q}{1 + [(4J_3 + 1)_p]_+} W_- \tag{45}$$

$$[B, V_+] = \frac{[2J_3]_q[(2J_3 - 1)_p]_+ - [(2J_3)_p]_+[2J_3 - 1]_q}{1 + [(4J_3 - 1)_p]_+} V_+ \tag{46}$$

$$[B, V_-] = -\frac{[2J_3]_q[(2J_3 + 1)_p]_+ - [(2J_3)_p]_+[2J_3 + 1]_q}{1 + [(4J_3 + 1)_p]_+} V_- \tag{47}$$

$$[B, W_+] = -\frac{[2J_3]_q[(2J_3 - 1)_p]_+ - [(2J_3)_p]_+[2J_3 - 1]_q}{1 + [(4J_3 - 1)_p]_+} W_+ \tag{48}$$

$$[B, W_-] = \frac{[2J_3]_q[(2J_3 + 1)_p]_+ - [(2J_3)_p]_+[2J_3 + 1]_q}{1 + [(4J_3 + 1)_p]_+} W_- \tag{49}$$

$$\begin{aligned}[J_+, J_-] = \frac{1}{f(p, q)g(p, q)} \{&\{B - [J_3]_{p,q} + 2[J_3 + 1/2]_{p,q}\}W_-V_+ \\ &+ \{-B - [J_3]_{p,q} + 2[J_3 + 1/2]_{p,q}\}V_-W_+ \\ &- \{B - [J_3]_{p,q} + 2[J_3 - 1/2]_{p,q}\}V_+W_- \\ &+ \{B + [J_3]_{p,q} - 2[J_3 - 1/2]_{p,q}\}W_+V_-\}\end{aligned} \tag{50}$$

Now when $q \to 1$ and $p \to 1$ equation (42) becomes

$$[J_+, V_-] = \frac{1}{f(1, 1)} V_+$$

but $[J_+, V_-] = V_+$; then $f(1, 1) = 1$. Also equation (43) becomes

$$[J_-, V_+] = \frac{-1}{g(1, 1)}$$

but $[J_-, V_+] = V_-$; we deduce that $g(1, 1) = -1$.

When we choose $f(p, q)$ and $g(p, q)$ as

$$f(p, q) = \frac{2}{p^{1/2} + q^{-1/2}}, \qquad g(p, q) = \frac{-2}{p^{1/2} + q^{-1/2}} \tag{51}$$

then we have

$$[J_+, V_-] = (p^{1/2} + q^{-1/2}) \frac{[2J_3]_q[(2J_3 - 1)_p]_+ - [(2J_3)_p]_+[2J_3 - 1]_q}{1 + [(4J_3 - 1)_p]_+} V_+ \tag{52}$$

$$[J_-, V_+] = -(p^{1/2} + q^{-1/2}) \frac{[2J_3]_q[(2J_3 + 1)_p]_+ - [(2J_3)_p]_+[2J_3 + 1]_q}{1 + [(4J_3 + 1)_p]_+} V_- \tag{53}$$

$$[J_+, W_-] = (p^{1/2} + q^{-1/2}) \frac{[2J_3]_q[(2J_3 - 1)_p]_+ - [(2J_3)_p]_+[2J_3 - 1]_q}{1 + [(4J_3 - 1)_p]_+} W_+ \tag{54}$$

$$[J_-, W_+] = -(p^{1/2} + q^{-1/2}) \frac{[2J_3]_q[(2J_3+1)_p]_+ - [(2J_3)_p]_+[2J_3+1]_q}{1 + [(4J_3+1)_p]_+} W_- \quad (55)$$

$$[B, V_+] = \frac{[2J_3]_q[(2J_3-1)_p]_+ - [(2J_3)_p]_+[2J_3-1]_q}{1 + [(4J_3-1)_p]_+} V_+ \quad (56)$$

$$[B, V_-] = -\frac{[2J_3]_q[(2J_3+1)_p]_+ - [(2J_3)_p]_+[2J_3+1]_q}{1 + [(4J_3+1)_p]_+} V_- \quad (57)$$

$$[B, W_+] = -\frac{[2J_3]_q[(2J_3-1)_p]_+ - [(2J_3)_p]_+[2J_3-1]_q}{1 + [(4J_3-1)_p]_+} W_+ \quad (58)$$

$$[B, W_-] = \frac{[2J_3]_q[(2J_3+1)_p]_+ - [(2J_3)_p]_+[2J_3+1]_q}{1 + [(4J_3+1)_p]_+} W_- \quad (59)$$

$$\begin{aligned}[J_+,\ J_-] = -\left(\frac{p^{1/2} + q^{-1/2}}{2}\right)^2 \{&\{B - [J_3]_{p,q} + 2[J_3 + 1/2]_{p,q}\} W_- V_+ \\ &+ \{-B - [J_3]_{p,q} + 2[J_3 + 1/2]_{p,q}\} V_- W_+ \\ &- \{B - [J_3]_{p,q} + 2[J_3 - 1/2]_{p,q}\} V_+ W_- \\ &+ \{B + [J_3]_{p,q} - 2[J_3 - 1/2]_{p,q}\} W_+ V_-\} \end{aligned} \quad (60)$$

In this paper we have extended the method of one-parameter quantum deformation (q-deformation) of Lie superalgebra to two-parameter quantum deformation with two parameters $p$, $q$ having no specific relation. We hope that it will be applicable for other types of Lie superalgebra and also for multiparameter quantum deformations.

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